Review of: Helmut Schwichtenberg Finite notations for infinite terms

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February 14, 2000

One of the aims of proof theory is to use the syntactical information given in a well formed formula or a well formed derivation to make analyses about what can and what cannot be done in certain formal systems. There are many ways to give such syntactical information. Some of the most important ones are

- Gentzen's system of natural deduction and of sequential calculi
- Kleene's system $\mathcal{O}$ of ordinal notation
- Veblen-Bachmann-Ackermann-Schütte-Takeuti systems of ordinal notation
- Martin-Löfs theory of types
- the Novikoff-Schütte-Tait analyses of arithmetic using ω-rules

The syntactical objects are either finite or infinite in a uniform way.

Wilfried Buchholz has in an important paper [1] showed how to give notation-systems for infinitary sequent-calculus derivations. An essential feature there is that from a notation one can find in a primitive recursive way information like "last rule used". Furthermore the terms are ordered in such a way that the syntactical transformation for getting cut-free derivations produces smaller terms.

The present paper does much the same thing for natural deduction systems (typed lambda-calculi) as Buchholz does for sequent-calculi. This turns out to be more complicated than first expected. The assignment goes in three steps. First we assign infinitary terms as in the Novikoff-Schütte-Tait analysis. Then we code them in a primitive recursive way using the uniformity in the infinitary terms. This is cumbersome, but more or less straightforward. The last step is new. Say we have a notation system $\mathcal{T}$ using primitive recursive codes for the infinitary terms. We extend $\mathcal{T}$ to a new notation system $\mathcal{T}^\star$ where we are also able to express directly that we have beta-conversion and reduction among terms. This is done by having new symbols for each variable, and function symbols for beta-conversion and for reduction.

The terms in $\mathcal{T}$ is ordered by the usual tree ordering of the infinitary terms. In $\mathcal{T}^\star$ this is extended to an ordering which also respects reductions among terms. This is done by a straightforward construction but where we get a jump up to the next epsilon-number.

So far about the notation systems. Now to the applications. First we consider the system PCF_α of partial recursive functionals with fixed point operator. In PCF_α we use approximation of functionals. Using the theory the author is able to prove in a direct and perspicuous way a trade-off theorem, where higher type recursion is traded off for simpler recursion over longer orderings. The second applications is to show continuous normalization for natural deduction. Both these applications are known. But they are notoriously hard to prove in a natural way. With the new proof here this is solved.

The author has shown how the new tools in proof theory given by Buchholz for sequential calculi can now also be used for systems of natural deduction (typed lambda-calculi).

References

[1] Wilfried Buchholz. *Notation systems for infinitary derivations.* Archive for Mathematical Logic, 30, pp 277-296. 1991.