Π^1_2-logic and ordinals

Dedicated to Jean-Yves Girard on the occasion of his 60th birthday

Herman Ruge Jervell

University of Oslo

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- J-Y G: Proof theory and logical complexity I. Bibliopolis 1987

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- Quantifiers analysed using eigen variables
- Subformula property
- Cut rule

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Up: Given a falsification of Γ, then by induction over the construction of TREE(Γ) we get a branch in TREE(Γ) with all formulas falsified and the branch is not secured.

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- Quantify over all not-secured branches

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- A node $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle$ in ORDINAL(α) is Γ-secured if the substitution tree TREE(Γ)$\alpha_0, \ldots, \alpha_{k-1} / o_0, \ldots, o_{k-1}$ is secured

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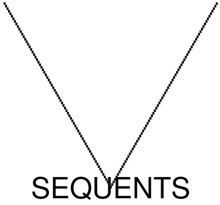
ORDINAL(α, Γ) is the tree we get from ORDINAL(α) by indicating for each node whether or not it is Γ-secured.



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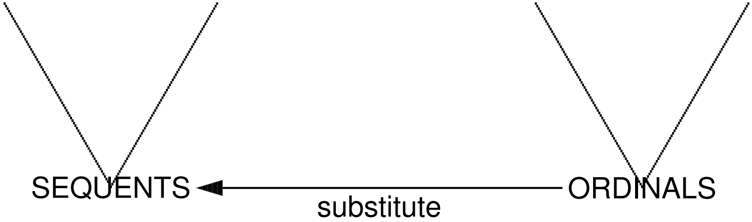

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- If ORDINALS(α, Γ) is not secured, then Γ is falsifiable with ordinals interpreted as $\beta \leq \alpha$
- ORDINALS(α, Γ) is secured for all α if and only if Γ is valid in β-logic

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For Γ to be valid in β-logic is a property of ORDINALS(ω, Γ)

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Strongly wellfounded 2

We need to consider extensions over all countable ordinals

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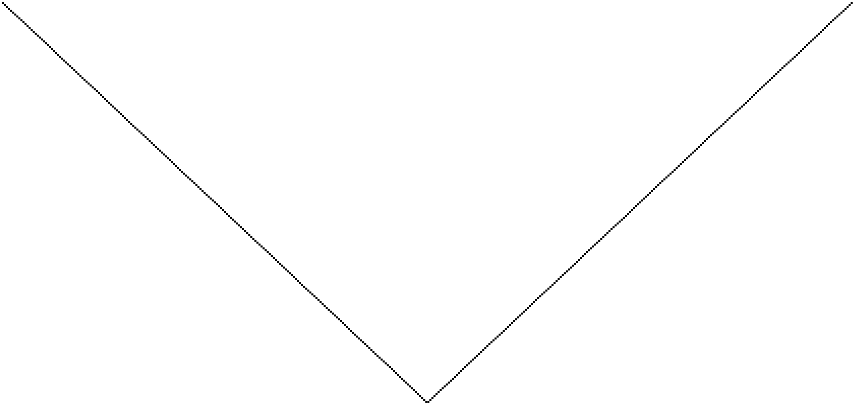
- G is homogeneous
- G$[\alpha]$ is well founded iff $\alpha < \gamma$

Dilators 1

The theory of strongly well founded homogeneous trees can be rephrased in a categorical framework. Note the following situation with $f \in I(\alpha, \beta)$ – an increasing function from α to β



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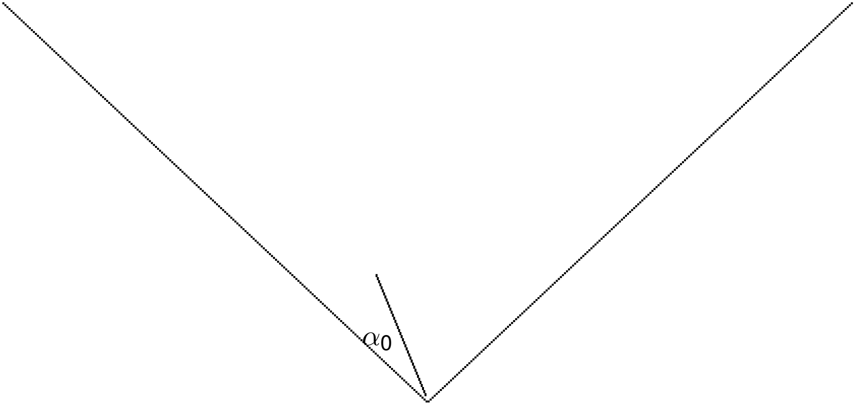

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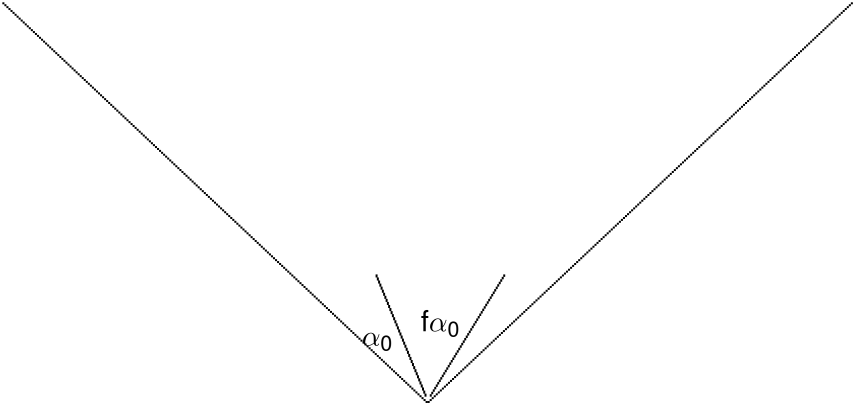

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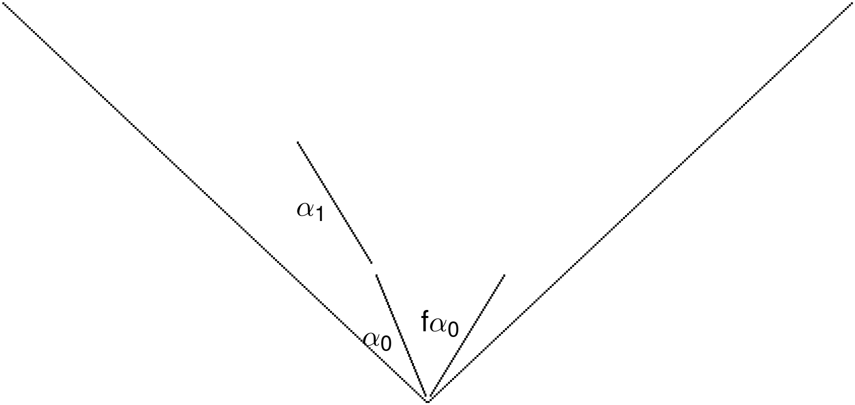

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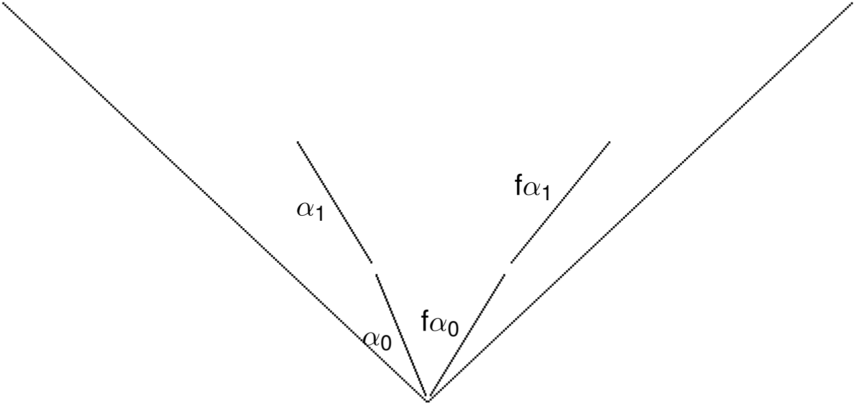

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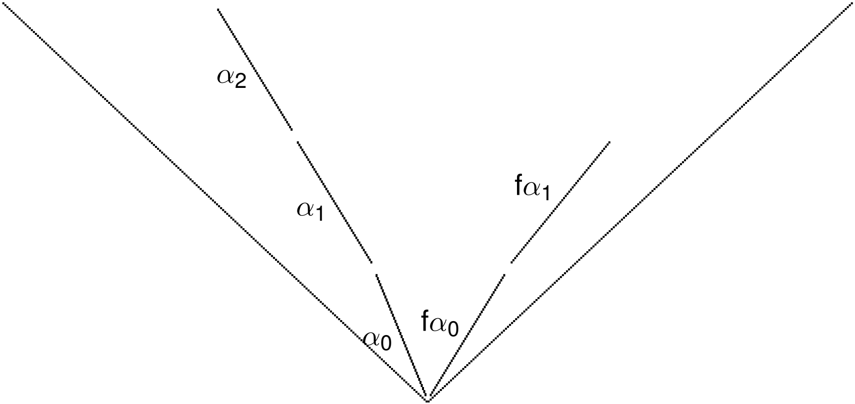

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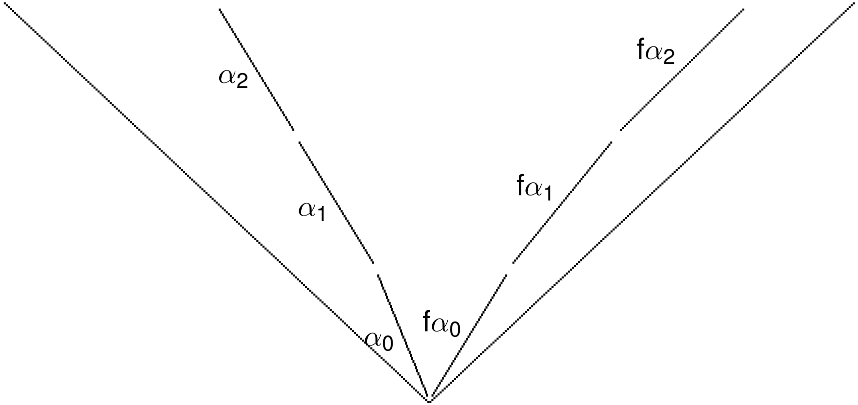

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- The corresponding dilator gives the ordinal function $\alpha \mapsto 2^\alpha$

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Below I will sketch a theory of ordinals

The first ordinals

We represent the finite numbers as unary trees

$$. = 0$$

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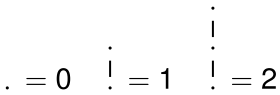

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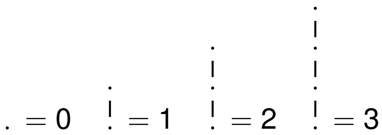

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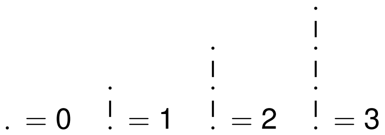

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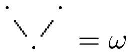

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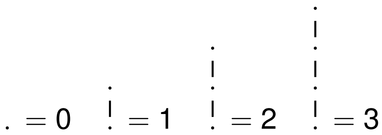

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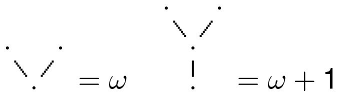

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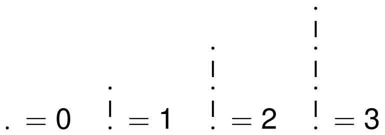

$= 0 \quad = 1 \quad = 2 \quad = 3$

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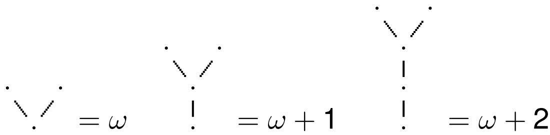

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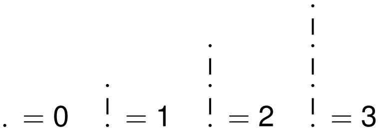

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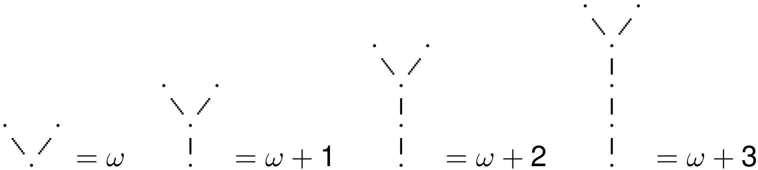

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$$S < T \Leftrightarrow S \leq \langle T \rangle \vee (\langle S \rangle < T \wedge \langle S \rangle < \langle T \rangle)$$

A decision tree

$\mathbf{A} < \mathbf{B}$ $\mathbf{B} < \mathbf{A}$

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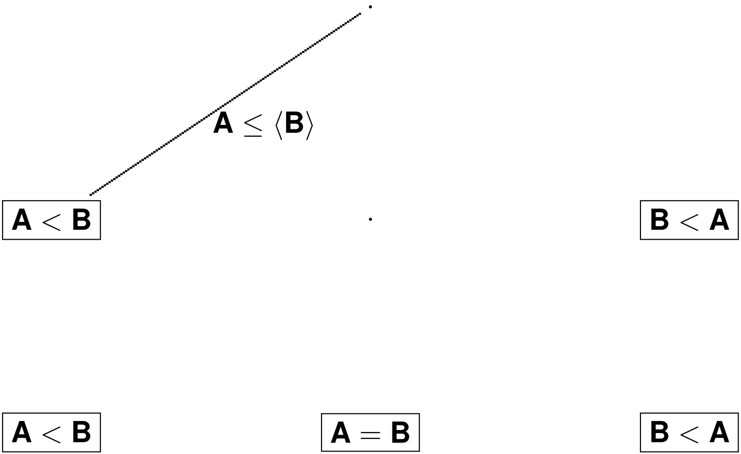

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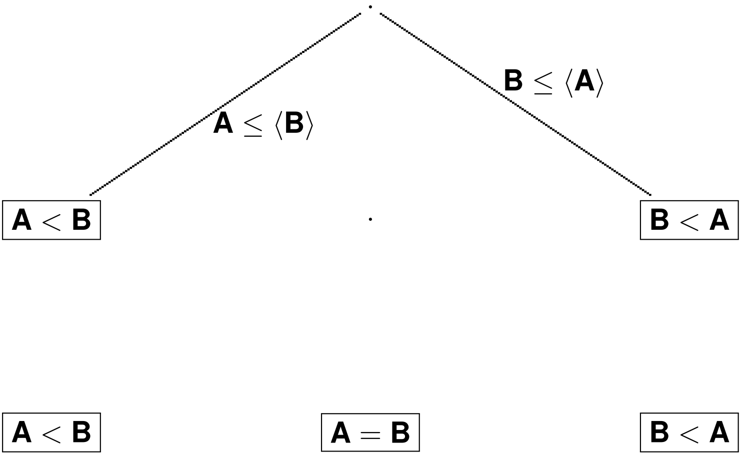

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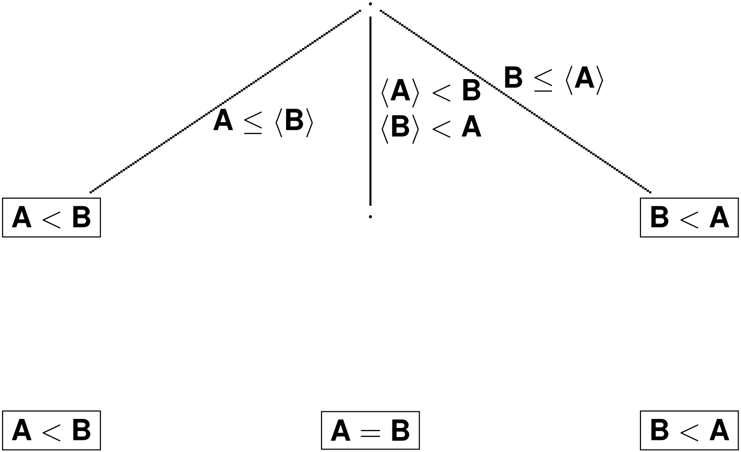

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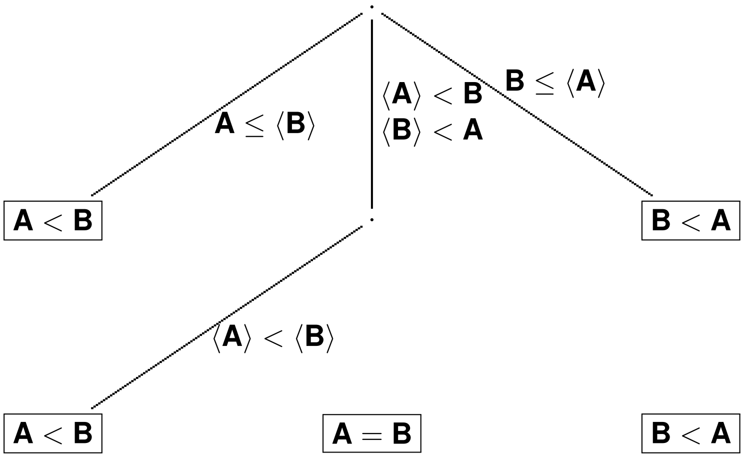

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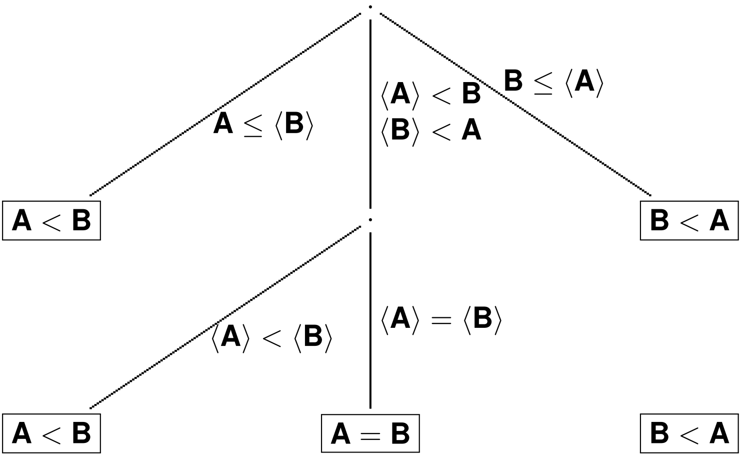

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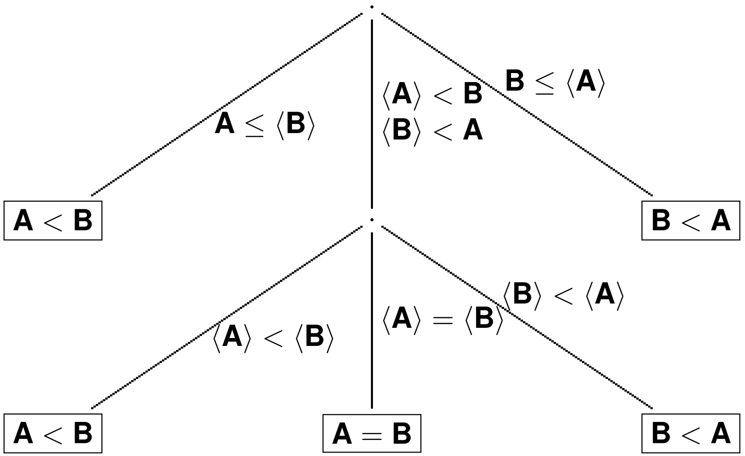

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- There is a 1-1 correspondence between the finite trees and an initial segment of the ordinals

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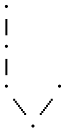 $= \omega \cdot 3$

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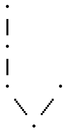
 $= \omega$  $= \omega \cdot 2$  $= \omega \cdot 3$

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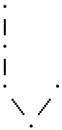
 $= \omega$  $= \omega \cdot 2$  $= \omega \cdot 3$

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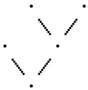
 $= \omega^{\omega}$  $= \omega^{\omega^{\omega}}$

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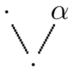
- 

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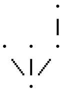
The Veblen function depends on how we start the function. It is too crude to have $x \mapsto \omega^{x}$ as an initial function. For more branchings in the tree we have Veblen functions with more arguments.

Approximations

Given the tree



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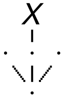
$X \mapsto$ 

Why Γ_0 is not special

In the usual definition of Veblen function we start with $x \mapsto \omega^x$ and then have Γ_0 as the limit of $x \mapsto \phi_x 0$. Then Γ_0 behaves sort of like ϵ_0. Instead of the binary trees going up to ϵ_0 we consider the ternary trees built up by

$$X, Y \mapsto$$

Levels of reasoning

- Σ^0_1
  - to be a finite tree
  - the successor: $S \mapsto \begin{array}{c} S \\ | \\ \cdot \end{array}$
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The finite trees give the ordinals up to the small Veblen ordinal — that is the ordinal connected with Kruskals theorem.

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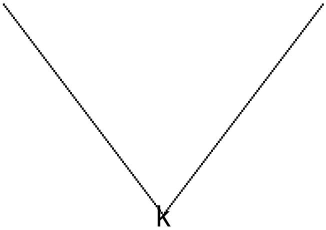

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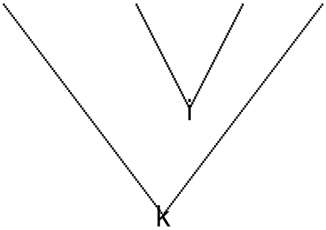

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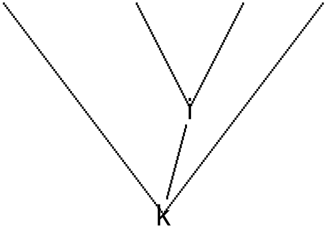

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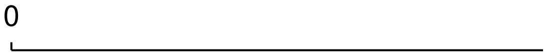
- If the labels are both i, then we take the lexicographical ordering of the immediate subtrees of S and T in the ordering $<_i$

The ordinal universe

The natural numbers

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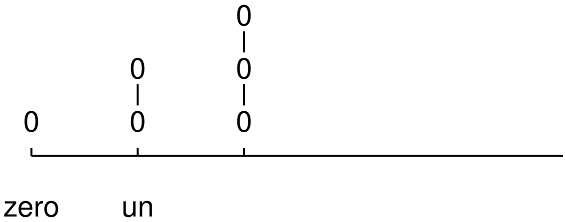

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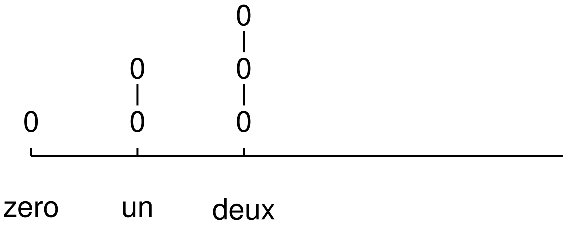

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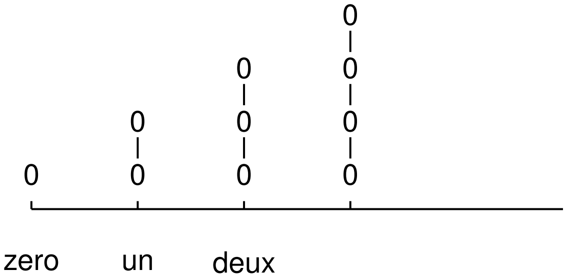
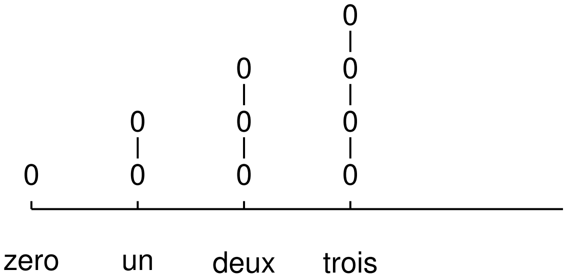
The natural numbers

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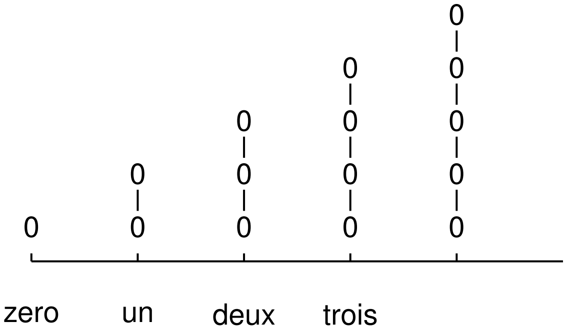
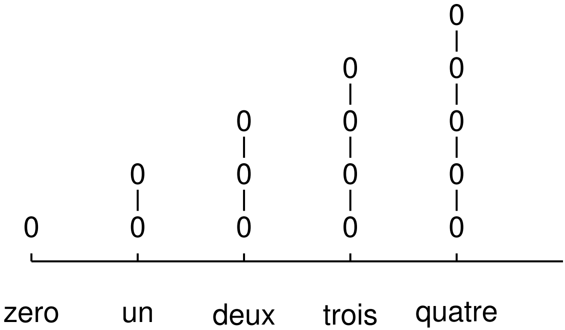
The ordinal universe
The natural numbers



The ordinal universe

The natural numbers



The ordinal universe

Up to ϵ_0

The ordinal universe

Up to ϵ_0

The ordinal universe

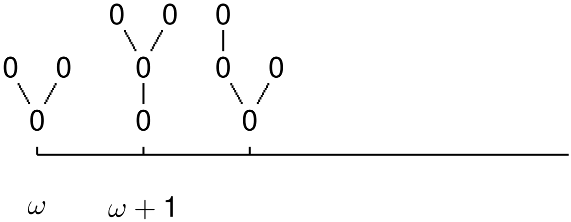
Up to ϵ_0

The ordinal universe

Up to ϵ_0

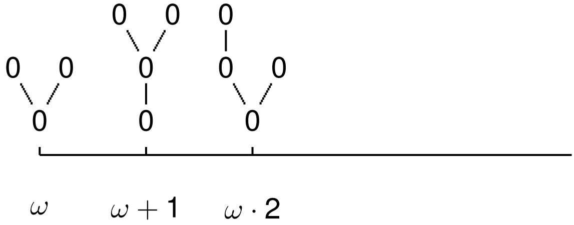
The ordinal universe

Up to ϵ_0



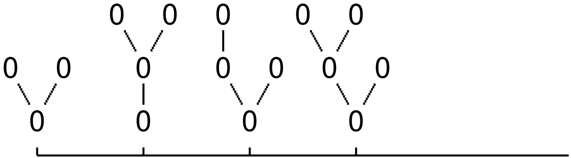
The ordinal universe

Up to ϵ_0



The ordinal universe

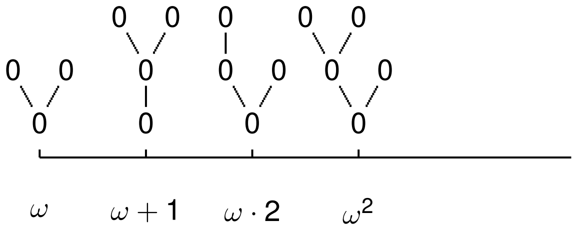
Up to ϵ_0



ω $\omega + 1$ $\omega \cdot 2$

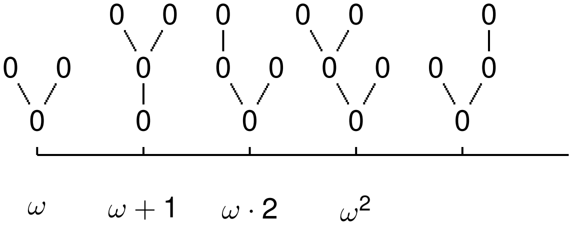
The ordinal universe

Up to ϵ_0



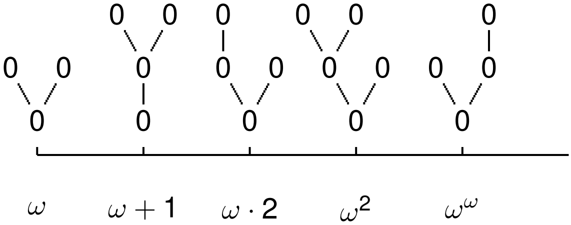
The ordinal universe

Up to ϵ_0



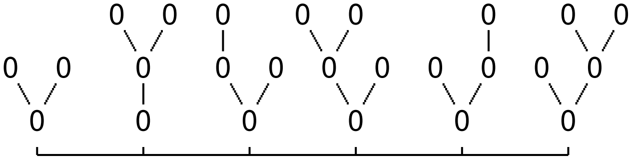
The ordinal universe

Up to ϵ_0



The ordinal universe

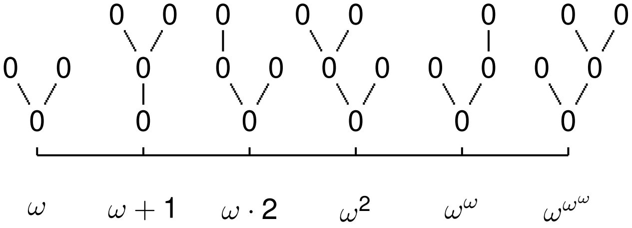
Up to ϵ_0



ω $\qquad \omega + 1$ $\qquad \omega \cdot 2$ $\qquad \omega^2$ $\qquad \omega^\omega$

The ordinal universe

Up to ϵ_0

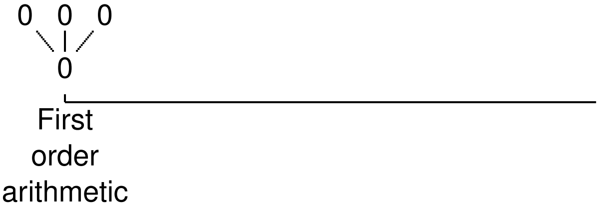


The ordinal universe

The predicative ordinals

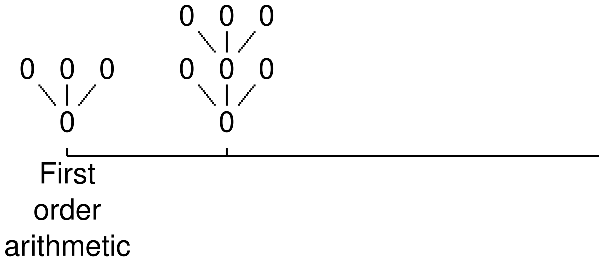
The ordinal universe

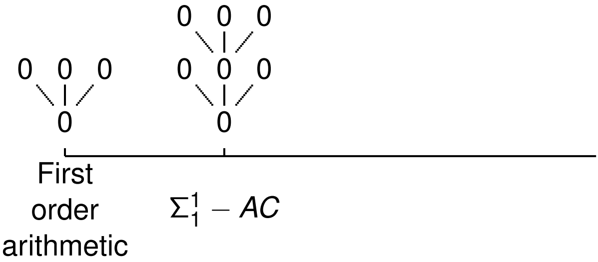
The predicative ordinals



The ordinal universe

The predicative ordinals

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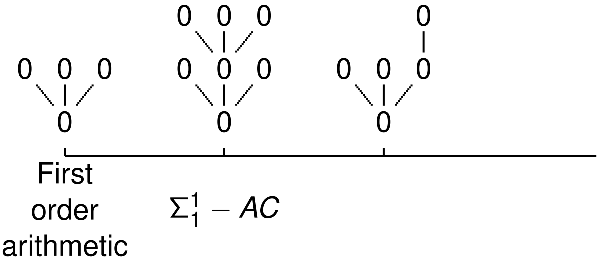
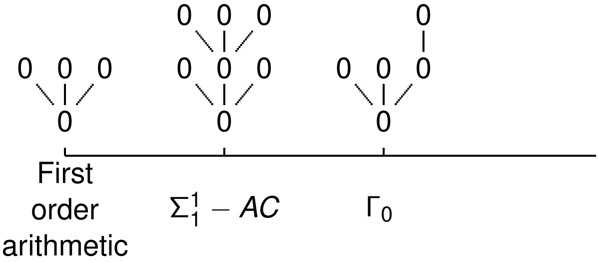
First order arithmetic
$\Sigma^1_1 - AC$


First order arithmetic
$\Sigma^1_1 - AC$

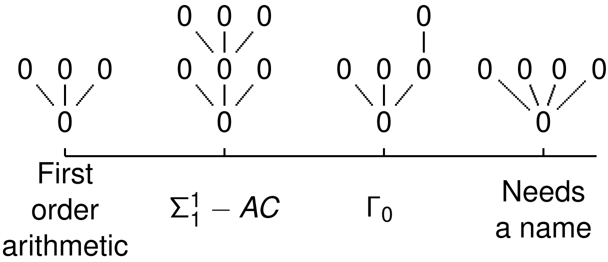
The ordinal universe

The predicative ordinals



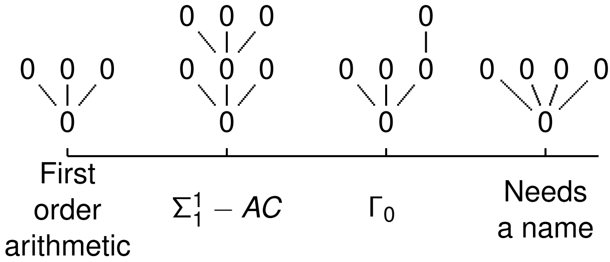
The ordinal universe

The predicative ordinals



The ordinal universe

The predicative ordinals



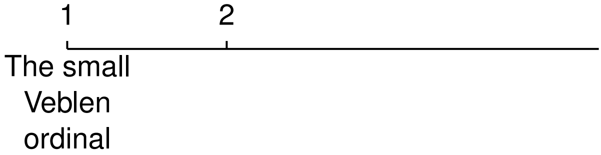

1

The ordinal universe

Beyond

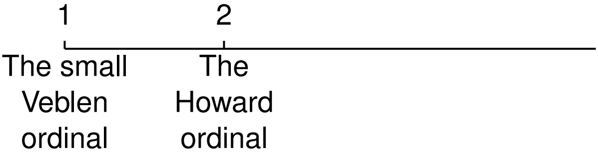
The ordinal universe

Beyond



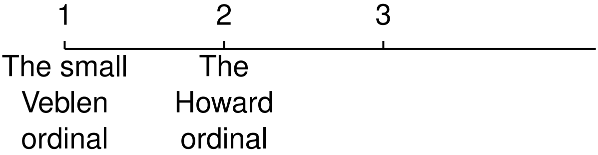
The ordinal universe

Beyond

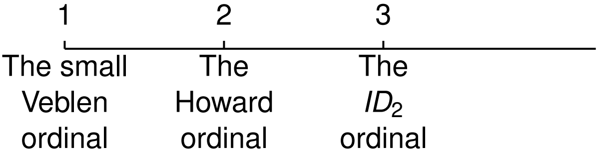


The ordinal universe

Beyond

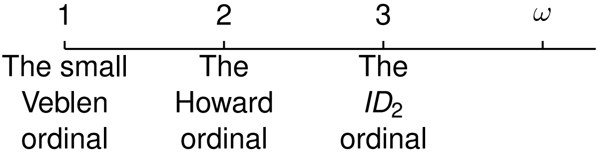


The ordinal universe
Beyond



The ordinal universe

Beyond



The ordinal universe

Beyond